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ON THE CONSTRUCTION OF SPLIT-FACE TOPOLOGIES

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ABSTRACT. We give a general theorem to facilitate the construction of interesting examples of split-face topologies of compact, convex sets.

Introduction. Let K be a compact convex set in a locally convex topological vector space. Let F be a closed face of K and F' be the union of all faces of K disjoint from F. It is always true that $K = \mathrm{co}(F \cup F')$ [1, Proposition II.6.5]. The face F is said to be *split* if F' is a face and K is the direct convex sum of F and F' [1, p. 133], i.e. if each $x \in K$ can be expressed by a unique convex combination

$$x = \lambda y + (1 - \lambda)z$$

with $0 \leq \lambda \leq 1$, $y \in F$ and $z \in F'$. The collection

$$\{F \cap \text{extreme points of } K \mid F \text{ closed split face of } K\}$$

forms the closed sets for a topology on the extreme points of K called the split-face or facial topology [1, p. 143]. Much is known about the split-face topology but there is a distinct lack of many interesting examples. This paper provides the first general results which may help to alleviate this problem.

We are much indebted to P. Taylor whose example (reproduced below) started us in the right direction.

Throughout, for a set $D \subseteq Y$ we let D^C be the complement of D in Y.

The construction. Let Y be a compact Hausdorff space and X a closed subset. Let $x \rightsquigarrow \rho_x$ be a weak* continuous map of X into $\{\mu \in C^*(Y)\colon \mu(\mathbf{1}) = 1\}$. Suppose X is divided into three disjoint pieces X_1, X_2, and X_3 with the following properties:

(1) For each $x \in X_2$, ρ_x is a probability measure.

(2) For each $x \in X$, $\rho_x \mid X_1 \cup X_2 \equiv 0$.

(3) For each $x \in X_3$, $\rho_x = \delta(x)$.

(4) $X_1 \cup X_2 \neq Y$.

Let $A = \{f \in C(Y)\colon f(x) = \rho_x(f) \text{ for all } x \in X\}$. We note that $\mathbf{1} \in A$. Let $\Phi\colon C^*(Y) \to A^*$ be the canonical map. For a set $F \subseteq Y$, we let

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$$P(F) = \{\mu \in C^*(Y): \mu \geq 0, \mu(1) = 1, \mu \text{ supported by } F\}.$$

Let $K = \Phi P(Y)$. We shall identify $y \in Y$ with $\Phi\delta(y) \in K$. Let ∂ be the Choquet boundary of A which, by our identification, is a subset of Y. Clearly, $\partial \subseteq Y - X_2$. We now make the further assumption:

(5) $\partial = Y - X_2$.

First, some obvious remarks.

Remark 1. X_1, X_2, and X_3 are Borel sets.

Indeed, it is trivial that X_3 is closed. As $\partial = Y - X_2$, we have that $X_2 \cup X_3 = \rho^{-1}(P(Y))$ and so $X_2 \cup X_3$ is closed. The remark is now clear.

Remark 2. There is an $M < \infty$ so that $\|\rho_x\| < M$ each $x \in X$.

Indeed, since X is compact and ρ is weak* continuous, the image of ρ is weak* compact and, so, norm bounded.

Theorem 1.

$$A^\perp = \left\{w \in C^*(Y): w(B) = w(B \cap X_1) + w(B \cap X_3) - \int_{X_1 \cup X_2} \rho_x(B)\, dw(x) \text{ for each Borel } B \subseteq Y\right\}.$$

Proof. The proof is contained in the proof of Theorem 1.2 of [5], which we reproduce here. We may assume $X_3 \neq \varnothing$ with no loss of generality (if it is empty, add a point of $Y - (X_1 \cup X_2)$ to X). Let

$$M = \{\delta(x) - \rho_x: x \in X_1 \cup X_2\}$$

and $X' = M \cup \{0\} = \{\delta(x) - \rho_x: x \in X\}$. As the map

$$x \to \delta(x) - \rho_x \tag{1}$$

is continuous, X' is weak* compact. The map (1) is clearly one-to-one from $X_1 \cup X_2$ onto M. Since X_3 is closed, both M and $X_1 \cup X_2$ are locally compact. Hence (1) is a proper map [2, Chapter 1, §10, No. 3, Corollary] and so $X_1 \cup X_2$ is homeomorphic to M [2, Chapter 1, §10, No. 1, Proposition 2].

Let Z be the weak* closed convex hull of X' and W its linear span. Clearly the weak* closure of W is $A^\perp$. Using [7, Proposition 1.2], we get

$$W = \left\{w \in C^*(Y): \text{There is a bounded regular Borel measure } \nu' \text{ on } M \text{ such that } w(f) = \int_M p(f)\, d\nu'(p) \text{ for each } f \in C(Y)\right\}.$$

Fix $w \in W$ and find the associated measure ν' on M. Using the homeomorphism between M and $X_1 \cup X_2$, ν' induces a measure υ on $X_1 \cup X_2$. Passing to finite sums and taking limits we get

$$\begin{aligned} w(f) &= \int_M p(f)\,d\nu'(p) \\ &= \int_{X_1 \cup X_2} (\delta(x) - \rho_x)(f)\,d\nu(x) \\ &= \nu(f) - \int_{X_1 \cup X_2} \rho_x(f)\,d\nu(x). \end{aligned}$$

But then we get [3, V, §3, Corollary to Proposition 12]

$$w(B) = \nu(B) - \int_{X_1 \cup X_2} \rho_x(B)\,d\nu(x)$$

for each Borel $B \subseteq Y$. In particular, for $B \subseteq X_1 \cup X_2$ we have $\rho_x(B) = 0$ and so $w(B) = \nu(B)$. Thus $w \mid X_1 \cup X_2 = \nu$ and so

$$W = \Big\{ w \in C^*(Y)\text{: For each Borel } B \subseteq Y,$$
$$w(B) = w(B \cap X_1) + w(B \cap X_2) - \int_{X_1 \cup X_2} \rho_x(B)\,dw(x) \Big\}.$$

Thus, to complete the proof, we need show that W is already weak* closed. By [4, V.5.9] it suffices to show W is norm closed. But this is clear using Remark 2 above.

Recall that for a measure μ on a nonempty compact set X in a lctvs E, $r(\mu) = x$ is the *resultant* of μ if for each continuous linear functional f on E we have $f(x) = \int f\,d\mu$.

Lemma 2. *With the identification of $y \in Y$ with $\Phi(\delta(y))$ in A^*, we have*:

(1) $K = \Phi P(\partial)$.

(2) *If $x \in C^*(Y)$ is the resultant of μ, a measure supported by* Ext(K), *then regarding μ as a measure on Y supported by ∂ we have $\Phi(\mu) = x = r(\mu)$.*

(3) *Let $\mu \in C^*(Y)$ and associate to it a measure μ' on ∂ by*

$$\mu'(B) = \mu(B \cap \partial) + \int_{X_2} \rho_x(B)\,d\mu(x).$$

Then $r(\mu') = \Phi(\mu)$.

Proof. (1) is just [7, Proposition 1.2]. For (2), let $f \in A =$ weak* linear functionals on K. Then

$$\begin{aligned} x(f) &= \int_{\mathrm{Ext}(K)} p(f)\,d\mu(p) = \int_\partial \delta(x)(f)\,d\mu(x) \\ &= \mu(f) = \Phi(\mu)(f). \end{aligned}$$

As for (3), let $f \in A$. Then

$$\begin{aligned} f(r(\mu')) &= \int_{\mathrm{Ext}(K)} p(f)\,d\mu'(p) = \mu'(f) \\ &= \int_\partial f(x)\,d\mu(x) + \int_{X_2} \rho_x(f)\,d\mu(x) = \int_{Y-X_2} f(x)\,d\mu(x) + \int_{X_2} f(x)\,d\mu(x) \end{aligned}$$

since $\rho_x = \delta(x)$ for $x \in X_2$ on A

$$= \mu(f) = \Phi(\mu)(f).$$

A set $D \subseteq Y$ is said to be *full* if the following conditions hold:

(1) For all $x \in X_1$, $(\delta(x) + \rho_x)(B) \neq 0$ for some $B \subseteq D$ implies that $\delta(x) + \rho_x$ is supported by D.

(2) For all $x \in X_2$, $x \in D$ implies ρ_x is supported by D.

Elementary properties of full sets are contained in the next lemma.

Lemma 3. (1) *If D is full, then for all $x \in X_1$, we have*:

$$(\delta(x) + \rho_x)(B) \neq 0 \quad \textit{for some } B \subseteq D^C$$
$$\Rightarrow \delta(x) + \rho_x \textit{ is supported by } D^C.$$

(2) *If D or D^C is full, then for all $x \in X_1$ we have*:

$$x \in D^C \Rightarrow \rho_x \mid D \equiv 0.$$

(3) *D is full iff the following two conditions hold*:

(a) *For all $x \in X_1$, $x \in D^C \Rightarrow \rho_x$ supported by D^C.*

(b) *For all $x \in X_1 \cup X_2$, $x \in D \Rightarrow \rho_x$ supported by D.*

Corollary 4. *Let D be full. Let μ be supported by $D \cap \partial$ and ν be supported by $D^C \cap \partial$. If $\Phi(\mu) = \Phi(\nu)$, then $\Phi(\mu) = 0 = \Phi(\nu)$. Hence, neither μ nor ν can be positive measures.*

Proof. Since $\Phi(\mu) = \Phi(\nu)$, we have $\mu - \nu \in A^\perp$. Thus, for any Borel $B \subseteq Y$ we have, by Theorem 1,

$$\begin{aligned}\mu(B) - \nu(B) = \mu(B \cap X_1) + \mu(B \cap X_2) - \int_{X_1 \cup X_2} \rho_x(B)\, d\mu - \nu(B \cap X_1) \\ - \nu(B \cap X_2) + \int_{X_1 \cup X_2} \rho_x(B)\, d\nu.\end{aligned} \tag{2}$$

Let $E \subseteq D$ be Borel. Then $\nu(E) = \nu(E \cap X_1) = \nu(E \cap X_2) = 0$. By Lemma 3(2), $\int_{X_1} \rho_x(E)\, d\nu = 0$. As ν is supported by $\partial = Y - X_2$, $\int_{X_2} \rho_x(E)\, d\nu = 0$. Hence, from (2),

$$\mu(E) = \mu(E \cap X_1) + \mu(E \cap X_2) - \int_{X_1 \cup X_2} \rho_x(E)\, d\mu. \tag{3}$$

Similar reasoning shows that for Borel $G \subseteq D^C$, $\mu(G) = \mu(G \cap X_1) = \mu(G \cap X_2) = \int_{X_1 \cup X_2} \rho_x(G)\, d\mu = 0$. Hence, (3) holds for each Borel set in Y and so $\mu \in A^\perp$. Hence, $\Phi(\mu) = 0 = \Phi(\nu)$. If μ were positive, say, then $\|\mu\| = \mu(\mathbf{1}) > 0$. As $\mathbf{1} \in A$, $\mu(\mathbf{1}) = \Phi(\mu)(\mathbf{1}) = 0$, a contradiction.

For ease of notation, for each $D \subseteq Y$ we let $T_D = \Phi P(D) = \{\Phi(\nu) \in A^*: \nu$ is a probability measure supported by $D\}$.

Proposition 5. *If D is full, then both T_D and $T_{\partial - D}$ are faces of K.*

Proof. As the map Φ is affine, both T_D and $T_{\partial-D}$ are convex subsets of K. Suppose $k \in T_D$ and $k = \alpha n + (1-\alpha)m$ with $n, m \in K$ and $0 < \alpha < 1$. Choose μ supported by $D \cap \partial$ and τ, λ in $P(\partial)$ so that $k = \Phi(\mu)$, $n = \Phi(\tau)$, and $m = \Phi(\lambda)$. In order to show T_D is a face, it clearly suffices to show that $\nu = \alpha\tau + (1-\alpha)\lambda$ is supported by $D \cap \partial$. Write $\nu = \nu|(D \cap \partial) + \nu|(D^C \cap \partial)$. Noting that $\Phi(\nu) = k = \Phi(\mu)$, we get $\Phi(\nu|(D^C \cap \partial)) = \Phi(\mu - \nu|(D \cap \partial))$. By Corollary 4, $\nu|D^C \cap \partial \equiv 0$ as it cannot be positive. Thus $\nu \in P(D \cap \partial)$ and so T_D is a face. A similar argument shows $T_{\partial-D}$ is a face.

We recall that for a face F of K, the union of all faces disjoint from F is denoted by F'. If face(k) denotes the minimal face of K containing k, then $F' = \bigcup\{\text{face}(k) \colon k \notin F\}$. To show that T_D is a split face for D closed and full, we must first describe T'_D. Toward this goal, we have the following lemma.

Lemma 6. *Suppose T_D is a face of K. Then $T'_D \subseteq T_{\partial \cap D^C}$.*

Proof. Let $k \in T'_D$. Then $k = \Phi(\mu)$ for μ a probability measure supported by ∂. We claim that $\mu(D) = 0$. Indeed, if not, then $\mu = \alpha(\mu \mid D)/\mu(D) + (1-\alpha)\nu$ where ν is either the zero measure or is supported by $\partial - D$. Then

$$k = \alpha\Phi\left(\frac{\mu|D}{\mu(D)}\right) + (1-\alpha)\Phi(\nu)$$

shows that $\Phi((\mu \mid D)/\mu(D)) \in \text{face}(k)$. As $(\mu \mid D)/\mu(D) \in P(D)$ we have $\Phi((\mu \mid D)/\mu(D)) \in \text{face}(k) \cap T_D = \varnothing$ since $k \in T'_D$. This contradiction establishes the claim and the lemma.

Theorem 7. *Let D be a closed full set. Then T_D is a closed split face of K with* Ext $T_D = D \cap \partial$ *and* $T'_D = T_{\partial \cap D^C}$.

Proof. The proof proceeds in three steps. We first claim that T_D is a closed face. From Proposition 5, we need only show that $T_D = \Phi P(D)$ is compact. As Φ is continuous and $P(D)$ compact, this is clear. Also clear are the facts Ext $T_D = D \cap \partial$ and $T_D = T_{D \cap \partial}$. We next claim that $T'_D = T_{\partial \cap D^C}$. From Lemma 6, we need only show that $T_{\partial \cap D^C} \subseteq T'_D$. Let $n \in T_D \cap T_{\partial \cap D^C}$. Then $n = \Phi(\mu) = \Phi(\nu)$ with $\mu \in P(D \cap \partial)$ and $\nu \in P(D^C \cap \partial)$. But this clearly cannot occur by Corollary 4 and so $T_D \cap T_{\partial \cap D^C} = \varnothing$. As $T_{\partial \cap D^C}$ is a face, $T_{\partial \cap D^C} \subseteq T'_D$.

Finally, we claim that T_D is split. Let $k \in K - (T_D \cup T'_D)$ and suppose we have two decompositions of k:

$$\begin{aligned} k &= \alpha_1 n_1 + (1-\alpha_1)m_1 \\ &= \alpha_2 n_2 + (1-\alpha_2)m_2 \end{aligned}$$

where $0 < \alpha_i < 1$, $n_i \in T_D$, and $m_i \in T'_D$. Find $\mu_i \in P(D \cap \partial)$ and $\nu_i \in P(D^C \cap \partial)$ so that $\Phi(\mu_i) = n_i$ and $\Phi(\nu_i) = m_i$. Then

$$\Phi(\alpha_1\mu_1 + (1-\alpha_1)\nu_1) = \Phi(\alpha_2\mu_2 + (1-\alpha_2)\nu_2)$$

and so

$$\Phi(\alpha_1\mu_1 - \alpha_2\mu_2) = \Phi((1 - \alpha_2)\nu_2 - (1 - \alpha_1)\nu_1).$$

From Corollary 4 we get

$$\Phi(\alpha_1\mu_1 - \alpha_2\mu_2) = 0 = \Phi((1 - \alpha_2)\nu_2 - (1 - \alpha_1)\nu_1). \tag{4}$$

Applying $\alpha_1\mu_1 - \alpha_2\mu_2$ to $\mathbf{1} \in A$ we get $\alpha_1 - \alpha_2 = 0$. Hence (4) yields $\Phi(\mu_1 - \mu_2) = 0$. Hence $n_1 = n_2$. Similarly $m_1 = m_2$ and the decompositions of k coincide.

We should now like to begin to prove a converse of Theorem 7.

Lemma 8. *Let $s \in X$ and suppose $\Phi(\tau) = \Phi(\rho_s)$ for some measure τ supported by $(X_1 \cup X_2)^C$. Then $\tau = \rho_s$.*

Proof. If $s \in X_3 \subset \partial$, then $\tau = \delta(s) = \rho_s$ clearly. So assume that $s \in X_1 \cup X_2$. Since $\tau - \delta(s) \in A^\perp$, by Theorem 1, for each Borel $B \subseteq Y$,

$$\begin{aligned}\tau(B) - \delta(s)(B) &= \tau(B \cap X_1) + \tau(B \cap X_2) - \int_{X_1 \cup X_2} \rho_x(B)\,d\tau - \delta(s)(B \cap X_1)\\ &\quad - \delta(s)(B \cap X_2) + \int_{X_1 \cup X_2} \rho_x(B)\,d\delta(s)(x)\\ &= \tau(B \cap X_1) + \tau(B \cap X_2) - \int_{X_1 \cup X_2} \rho_x(B)\,d\tau - \delta(s)(B) + \rho_s(B).\end{aligned}$$

As τ is supported by $(X_1 \cup X_2)^C$, we get

$$\tau(B) - \delta(s)(B) = -\delta(s)(B) + \rho_s(B)$$

and the result is immediate.

Theorem 9. *Let F be a closed split face of K. Let $D = \overline{F \cap \partial}$. Then $F = T_D$, $F' = T_{\partial \cap D^C}$, and D is full.*

Proof. Since $F \cap \partial = \operatorname{Ext} F$, we have $F = \overline{\operatorname{co}}(F \cap \partial)$ and so $F = T_{F\cap\partial} = T_D$. We claim that $F' = T_{\partial \cap D^C}$. Indeed, let $y \in \partial \cap D^C$. Then $\operatorname{face}(y) = \{y\}$ is disjoint from F and so $y \in F'$. Now let $\mu \in P(\partial \cap D^C) \subseteq P(F')$. Then [1, Corollary II.6.11] implies that $r(\mu) \in F'$. Hence $T_{\partial \cap D^C} \subseteq F'$. As Lemma 6 yields $F' = T'_D \subseteq T_{\partial \cap D^C}$ we have proven our claim.

To show that D is full, we will use the criteria of Lemma 3(3). Let $s \in X_1 \cup X_2$. Let $\nu_1 = \rho_s^+ \mid D$, $\nu_2 = \rho_s^+ | \partial \cap D^C, \nu_3 = \rho_s^- | D$, and $\nu_4 = \rho_s^- \mid \partial \cap D^C$. Let $\beta_i = \|\nu_i\|$ and $\mu_i = \nu_i/\|\nu_i\|$ (take $\mu_i = 0$ whenever $\|\nu_i\| = 0$). Then $\rho_s = \beta_1\mu_1 + \beta_2\mu_2 - \beta_3\mu_3 - \beta_4\mu_4$. Hence

$$\Phi(\delta(s) + \beta_3\mu_3 + \beta_4\mu_4) = \Phi(\beta_1\mu_1 + \beta_2\mu_2). \tag{5}$$

Applying these elements of A^* to $\mathbf{1} \in A$ we get

$$1 + \beta_3 + \beta_4 = \beta_1 + \beta_2.$$

Let β be this common number.

We first assume that $s \in F$. Then

$$s/(1+\beta_3)+\beta_3\Phi(\mu_3)/(1+\beta_3)=f\in F$$

and so

$$\frac{1+\beta_3}{\beta}f+\frac{\beta_4}{\beta}\Phi(\mu_4)=\frac{\beta_1}{\beta}\Phi(\mu_1)+\frac{\beta_2}{\beta}\Phi(\mu_2). \tag{6}$$

From the characterization of F' we know that $\Phi(\mu_2)$, $\Phi(\mu_4) \in F'$. Since f, $\Phi(\mu_1) \in F$ and F is split, (6) implies that $f = \Phi(\mu_1)$, $\Phi(\mu_4) = \Phi(\mu_2)$, and $1 + \beta_3 = \beta_1$. Plugging this into (5) yields

$$\Phi(\delta(s)+\beta_3\mu_3)=\Phi(\beta_1\mu_1)$$

and so $\Phi(\beta_1\mu_1 - \beta_3\mu_3) = \Phi(\rho_s)$. Since $\beta_1\mu_1 - \beta_3\mu_3$ is supported by $(X_1 \cup X_2)^C$, Lemma 8 implies that $\beta_1\mu_1 - \beta_3\mu_3 = \rho_s$. Hence ρ_s is supported by D. So we have shown that $s \in D \cap (X_1 \cup X_2)$ implies ρ_s is supported by D. Similarly, for $s \in D^C \cap X_1 \subseteq \partial \cap D^C \subseteq F'$, we get ρ_s is supported by $\partial - D \subseteq D^C$. Hence all the criteria of Lemma 3(3) have been verified.

We have now arrived at our characterization of facially closed sets. Recall that a set $D \subseteq \operatorname{Ext} K$ is facially closed if there is a closed split face F with $\operatorname{Ext} F = D$.

Theorem 10. *Let $D \subseteq Y - X_2$. Then D is facially closed if and only if the following conditions hold*:

(a) *D is closed in $Y - X_2$.*

(b) *$\overline{D}$ is full.*

Proof. Suppose conditions (a) and (b) hold. From (b) and Theorem 7, we get $T_{\overline{D}}$ is a closed split face and $\operatorname{Ext} T_{\overline{D}} = \overline{D} \cap \partial$. From (a), $\overline{D} \cap \partial = D$ and so D is facially closed. Conversely, suppose D is facially closed. As all facially closed sets are closed in $\operatorname{Ext} K = \partial$, we get (a). Also, there is a closed split face F with $F \cap \partial = D$. Theorem 9 now yields (b).

As an example of Theorem 10 we cite the following construction. Suppose X, Y, ∂ and ρ are as above. Suppose further:

(a) $X_2 = \varnothing$.

(b) ρ_s is a finite linear combination of point masses in X_1^C for each $s \in X_1$.

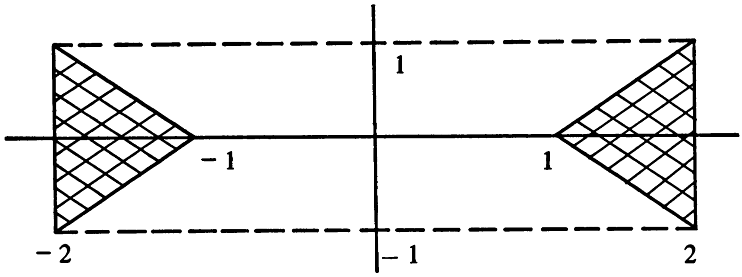
(c) For $s \neq s'$ elements of X_1, $\operatorname{supp}\rho_s \cap \operatorname{supp}\rho_s' = \varnothing$.

On Y define an equivalence relation $\sim$ by the following: $p \sim q$ means that there exists an $s \in X_1$ with both p and q in $\operatorname{supp}\rho_s \cup \{s\}$.

Theorem 11. *Assume the above hold. Then the factor topology on $Y/\sim$ "is" the facial topology on $Y - X_2$ and it is always T_1.*

Proof. A set $F \subseteq Y/\sim$ is closed iff F is closed in Y and F is saturated for the relation $\sim$. Note that F is saturated for the relation $\sim$ iff for every $s \in X_1$, $(\operatorname{supp}\rho_s \cup \{s\}) \cap F \neq \varnothing$ implies $\operatorname{supp}\rho_s \cup \{s\} \subseteq F$. But the latter statement is precisely the statement that F is full. Thus, by Theorem 10, $F \subseteq Y/\sim$ is closed iff F is facially closed in Y.

Example 1 (P. Taylor). Let Y be the shaded region of the plane in the following diagram:



We take $X = \{(x+1,y)\colon 0 \leq y \leq x, 0 \leq x \leq 1\}$ and define for $s \in X$:

$$\rho_s = \delta(-x-1,-y) + \delta(x+1,-y) - \delta(-x-1,y).$$

Clearly $s \rightsquigarrow \rho_s$ is weak* continuous on X and $\rho_s(1) = 1$. We have $X_2 = \varnothing$ and $X_3 = \{(x+1,y) \in X\colon y = 0\}$, with $X_1 = X - X_3$. Let

$$A = \{f \in C(Y)\colon f(s) = \rho_s(f) \text{ for all } s \in X\}.$$

Then clearly each $y \in Y$ is a peak point for A, i.e. for all $y \in Y$, there is an $f \in A$ with $f(y) = 1 > |f(p)|$ for $p \neq y$. Hence $\partial = Y$ and so all five of our assumptions are fulfilled. We define $\sim$ on Y as we did above: $p \sim q$ iff there is an $s \in X$ with $p, q \in \operatorname{supp} \rho_s \cup \{s\}$. Clearly each element of the decomposition under $\sim$ is of the form $\{(x+1,y),\ (x+1,-y),\ (-x-1,y),\ (-x-1,-y)\}$ for $0 \leq y \leq x$, $0 \leq x \leq 1$. It is easily verified that the projection map $\pi\colon Y \to Y/\sim$ is open so the factor space $Y/\sim$ with the factor topology is first countable, second countable, compact, and locally compact though not Hausdorff. Via Theorem 11, the facial topology on Y is first countable, second countable, compact, and locally compact though not Hausdorff. In [6], we studied the facial topology (among other topologies on the extreme points of compact convex sets). We proved there that if (Y, facial topology) satisfied an auxiliary condition (C2), then the properties of first countability, second countability, and local compactness for (Y, facial topology) were equivalent. We did not know then whether (C2) was necessary for the conclusion. The above construction provides the necessary example for it does not satisfy even (C1), a weaker condition than (C2).

Proposition 12. *There is a compact metrizable convex set K with closed extreme points whose facial topology is first countable, second countable and locally compact but which does not satisfy*:

(C1) *If $\{p_n\} \subseteq \operatorname{Ext}(K)$ converges to q and if $\{p_n\}$ converges to p in the facial topology, then p belongs to the minimal closed split face containing q.*

Proof. We take K to be the set $\{\mu \in A^*: \mu(1) = 1 = \|\mu\|\}$. Then $\mathrm{Ext}(K) = Y$. We take $p_n = (1 + 1/n, 1/n)$, $q = (1,0)$, and $p = (-1,0)$. Clearly $p_n \to q$ and $\{p_n\}$ converges to both p and q in the facial topology. Finally, the minimal closed split face containing q is $\{q\}$ which clearly does not contain p.

Example 2. (Rogalski in [8]). Let dx be Lebesgue measure on [0, 1] and μ be the measure $\sum_{n=1}^{\infty} 2^{-n}\delta(x_n)$ where $\{x_n\}$ is an enumeration of the rational numbers in [0, 1]. We take $X_1 = X_3 = \varnothing$, $X_2 = \{x_1\}$ and

$$\rho_{x_1} = 2dx - 2\sum_{n=2}^{\infty} 2^{-n}\delta(x_n).$$

Then

$$\begin{aligned} A &= \{f \in C[0,1] \mid f(x) = \rho_x(f) \text{ all } x \in X\} \\ &= \left\{f \in C[0,1] \mid \sum 2^{-n} f(x_n) = \int f dx\right\}. \end{aligned}$$

All the assumptions regarding the map ρ are trivial in this case and [8, Proposition 20] shows that $\partial = [0,1]$ so all of our results above apply. Thus, by Theorem 10, a set $D \subseteq [0,1]$ is facially closed iff D is a closed subset of the irrational numbers in [0, 1] or $D = [0,1]$. Hence, each irrational number in [0, 1] is a split face (a fact already established in [8, Corollary 25]) but no rational number in [0, 1] is a split face.

Proposition 13. *There exists a compact convex set K whose extreme points E form a closed set for which the collection of extreme points which are split faces are dense in E and for which the collection of extreme points which are not split faces are dense in E.*

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